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APPROXIMATIONS AND THE SPECTRAL PROPERTIES OF MEASURE-PRESERVING GROUP ACTIONS

BY

G. W. RILEY

ABSTRACT

This paper presents some extensions and applications of the method of approximations of ergodic theory (see [6]). Two notions of approximation are defined which are applicable to arbitrary σ-finite-measure-preserving group actions (see §1). Building upon results of [2], [13] and [6], the speeds of such approximations are related to the questions of spectral multiplicity, spectral type and ergodicity (see §3). For the result on spectral multiplicity, there is first established a general result concerning the spectral decomposition of unitary representations (see §2). The last section is devoted to applications — chiefly to certain classes of cylinder transformations which arise in connection with irregularity of distribution (see [12]). These transformations provide examples (on infinite measure spaces) of approximations of all finite multiplicities. The method of approximations is shown to be a natural tool for the study of their spectral properties.

Introduction

In their fundamental paper [6], Katok and Stepin introduced the "method of approximations" into ergodic theory. In so doing they showed that many properties of a measure-preserving automorphism of a probability space may be deduced from the existence of certain types of approximation by periodic transformations.

We shall present some extensions and applications of those aspects of the method of approximations which deal with the determination of spectral properties. Our results build upon the developments which have already been achieved by Chacon [2] and Stepin [13].

We begin, in the next section, by defining two notions of approximation, each of which is applicable to arbitrary measure-preserving group actions.

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Our main general result, Theorem 3.1, relates the spectral multiplicity of a measure-preserving action of a type I group to the speed with which it admits finite multiplicity approximations. The key to the proof of this theorem is Proposition 2.1, a result in the spectral decomposition theory of unitary group representations.

Turning to the link between cyclic approximations and the questions of singularity of spectral type and ergodicity, we have included the statements of two theorems, 3.4 and 3.5, which place in our more general context earlier results of [13] and [6].

The final section is devoted to applications, including an investigation of the spectral properties of the class of cylinder transformations $\{T_{\alpha,\beta} : \alpha, \beta \in (0,1)\}$, defined on $[0,1) \times \mathbf{R}$ as follows:

$$T_{\alpha,\beta}(x,t) = (x+\alpha \ (\mathrm{mod}\ 1), t + \chi_{[0,\beta)}(x) - \beta), \qquad \text{for all } (x,t) \in [0,1) \times \mathbf{R}.$$

These transformations, which arise naturally in connection with the irregularity of distribution of the sequences $n\alpha$, $n = 0, 1, 2, \cdots$, α irrational, have been studied by a number of authors [1, 3, 8, 11, 12]. However, it seems that, until now, only the question of their ergodicity has received attention. In §4, conditions on α and β are given under which the transformation $T_{\alpha,\beta}$ has simple, singular spectrum. There also appears a class of cylinder transformations each of which has spectral multiplicity uniformly equal to two (see Corollary 4.10).

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§1. Finite multiplicity and cyclic approximations

Let X be a standard Borel space equipped with a σ-finite measure μ.

DEFINITION 1.1 (cf. del Junco [5]). (a) A semi-partition (of X) is a countable collection of pairwise-disjoint, non-null, measurable subsets of X.

(b) A partition (of X) is a semi-partition whose elements cover all of X.

(c) Two semi-partitions ξ and ξ' shall be described as disjoint if $(\bigcup_{C \in \xi} C)$ and $(\bigcup_{C' \in \xi'} C')$ are disjoint subsets of X. In this case, $\xi \vee \xi'$ shall denote the semi-partition which contains all the elements of both ξ and ξ'.

DEFINITION 1.2. A sequence of semi-partitions $\xi(n)$, $n = 1,2,\cdots$ is said to converge to the unit partition, denoted $\xi(n)\to\varepsilon$, if, whenever A is a measurable subset of X of finite measure, then

$$\lim_{n\to\infty}\mu\,(A\,\Delta A\,(\xi(n)))=0,$$

for some choice of successive approximations to A by unions, $A(\xi(n))$, of the elements of $\xi(n)$, $n = 1,2,\cdots$.

Let there be given a measure (μ)-preserving, Borel action, $(g,x)\to g\cdot x$, $g\in G$, $x\in X$, of a locally compact second countable group G on the space X.

In the definitions that follow, N denotes a positive integer and $f(n)$, $n = 1,2,\cdots$ a sequence of non-negative real numbers.

DEFINITION 1.3 (cf. definition 2.4 of [2]). We shall say that a measure-preserving action of G on X admits a multiplicity N approximation with speed $f(n)$ if there exists a sequence of finite semi-partitions $\xi(n)$, $n = 1,2,\cdots$ such that

(i) $\xi(n)\to\varepsilon$, as $n\to\infty$; and such that, for each n, there is a splitting

$$\xi(n)=\xi_1(n)\vee\xi_2(n)\vee\cdots\vee\xi_N(n)$$

into N mutually disjoint component semi-partitions

$$\xi_j(n)=\{C_{ij}(n): i=1,\cdots,q_j(n)\},\qquad j=1,\cdots,N,$$

for each of which

(ii) all of the elements are of the same finite measure, i.e. $\mu(C_{ij}(n))=\mu(C_{1j}(n))<\infty$, for all $i=1,\cdots,q_j(n)$, and

(iii) group elements $g_{1j}(n),\cdots,g_{q_j(n)-1,j}(n)$ may be chosen so that

$$(1/(q_j(n)\mu(C_{1j}(n))))\cdot\sum_{i=1}^{q_j(n)-1}\mu\,(g_{ij}(n)\cdot C_{ij}(n)\Delta C_{i+1,j}(n))\leqq f(q_j(n)).$$

The following defines a special type of multiplicity one approximation.

DEFINITION 1.4 (cf. definition 1.1 of [6]). We shall say that a measure-preserving action of G on X admits a cyclic approximation with speed $f(n)$ if there exists a sequence of finite semi-partitions

$$\xi(n)=\{C_1(n),\cdots,C_{q(n)}(n)\},\qquad n=1,2,\cdots,$$

such that

(i) $\xi(n) \to \varepsilon$, as $n \to \infty$,

(ii) in each $\xi(n)$, all of the elements are of the same finite measure, i.e. $\mu(C_i(n)) = \mu(C_1(n)) < \infty$, for all $i = 1, \cdots, q(n)$, and

(iii) for each $n = 1, 2, \cdots$, group elements $g_1(n), g_2(n), \cdots, g_{q(n)}(n)$ may be chosen so that

$$(1/(q(n)\mu(C_1(n)))) \cdot \sum_{i=1}^{q(n)} \mu(g_i(n) \cdot C_i(n) \Delta C_{i+1}(n)) \leqq f(q(n)),$$

where $C_{q(n)+1}(n)$ is taken to denote $C_1(n)$.

Remark 1.5. The **Z**-action generated by an irrational rotation of the circle admits cyclic approximations with arbitrary speed. In terms of definition 1.1 of [6], this need not be true of the irrational rotation itself.

For **Z**-actions, apart from the dropping of the customary hypothesis that μ be a probability measure, the novel feature of Definitions 1.3 and 1.4 is the free choice of group elements which condition (iii) of each of them allows.

Remark 1.6. Conditions (i) and (ii) of Definition 1.4 imply that

$$\lim_{n \to \infty} q(n)\mu(C_1(n)) = \mu(X).$$

If μ is finite, then the normalizing factor $1/(q(n)\mu(C_1(n)))$ in condition (iii) could be replaced by $1/\mu(X)$ without affecting subsequent results.

§2. Some spectral multiplicity theory

If G is a type I group, then any continuous unitary representation Π of G on a separable Hilbert space admits a canonical spectral decomposition

$$(*) \qquad \Pi \cong \int_{\hat{G}}^{\oplus} m(\lambda) \cdot \lambda d\nu(\lambda),$$

where

(a) $\hat{G}$ denotes the standard Borel space of unitary equivalence classes of irreducible representations of G,

(b) m is a measurable function on $\hat{G}$ with values in $\{\infty; 1, 2, \cdots\}$, known as the spectral multiplicity function of the representation Π,

(c) $m(\lambda)\cdot\lambda$ denotes the direct sum of $m(\lambda)$ copies of the irreducible λ, for all $\lambda \in \hat{G}$, and

(d) ν is a σ-finite, Borel measure on $\hat{G}$, known as a spectral measure of Π.

Note that (*) determines the pair (ν, m) uniquely up to equivalence of measures and almost everywhere equality of functions. The equivalence class of the measure ν is called the maximal spectral type of the representation Π.

For more details of the above spectral decomposition, see Dixmier [4] or Mackey [10].

When the terms spectral multiplicity, spectral type, etc., are applied to a measure (μ)-preserving group action, $(g, x) \to g \cdot x$, $g \in G$, $x \in X$, they are understood to refer to the corresponding spectral properties of the unitary representation, Π_g, $g \in G$, induced on $L^2(X, \mu)$ as follows:

$$(\Pi_g y)(x) = y(g^{-1}\cdot x), \quad \text{for all } y \in L^2(X,\mu) \text{ and } x \in X.$$

See Kirillov [7].

At this point, it is convenient to introduce the following notation: given a type I group G, then, for each $k \in \{\infty; 1, 2, \cdots\}$, $\hat{G}_k$ denotes the Borel subset of $\hat{G}$ consisting of all elements λ with $\dim \lambda = k$; given a representation Π defined on a space H, we shall denote by $Z(z)$ the closed, Π-cyclic subspace generated by a vector z in H.

The following proposition was proved by Chacon [2] in the case of a single unitary operator (i.e. for $G = \mathbf{Z}$). The main innovation in our extension of Chacon's proof to cover the case of a general type I group is Lemma 2.2, below.

Proposition 2.1. *Let Π be a continuous, unitary representation of a type I group G on a separable Hilbert space H. Fix a positive integer k. Suppose that l is any positive integer chosen so that m, the spectral multiplicity function of Π, is greater than or equal to l on a non-null subset of $\hat{G}_k$ (with respect to the maximal spectral type of Π).*

Then there exist l orthonormal vectors $y_1, \cdots, y_l$ in H such that

$$\sum_{j=1}^{l} d^2(y_j, Z(z)) \geqq l - k,$$

for all $z \in H$, where d denotes the distance induced from the norm on H.

The following lemma may be seen to deal with the case when the maximal spectral type of Π is concentrated at a single point.

LEMMA 2.2. *Suppose that* $H = H_k \otimes H_l$, *where* H_k *and* H_l *are fixed Hilbert spaces of finite dimensions* k *and* l *respectively. Let* $\{w_1, w_2, \cdots, w_l\}$ *be an orthonormal basis for* H_l, *and choose arbitrary unit vectors* $v_1, \cdots, v_l$ *in* H_k.

Then, whenever Π *is a unitary representation of* G *on* H *of the form* $\lambda \otimes \mathrm{Id}_{H_l}$, *with* $\lambda \in \hat{G}_k$, *one has*

$$\sum_{j=1}^{l} d^2(v_j \otimes w_j, Z(z)) \geqq l - k,$$

for all $z \in H$.

PROOF. Choose an orthonormal basis $\{u_1, \cdots, u_k\}$ for H_k. The tensor product space H splits into the orthogonal direct sum of the k subspaces $\{u_i \otimes w ; w \in H_l\}$, $i = 1, \cdots, k$. Hence, each vector z in H has a unique decomposition as a sum

$$z = \sum_{i=1}^{k} u_i \otimes z_i, \qquad \text{with } z_i \in H_l, \quad \text{for } i = 1, \cdots, k.$$

So, fixing an arbitrary vector z in H, one may write

$$\Pi_g z = \sum_{i=1}^{k} (\lambda_g u_i) \otimes z_i, \quad \text{for all } g \in G.$$

From this expression and the irreducibility of λ, it follows that

$$Z(z) = H_k \otimes \text{lin. span}\{z_1, \cdots, z_k\}.$$

Hence, the orthogonal projection with range $Z(z)$ is just $\mathrm{Id}_{H_k} \otimes P$, where P denotes the projection from H_l onto the subspace spanned by $z_1, \cdots, z_k$.

Now,

$$\sum_{j=1}^{l} d^2(v_j \otimes w_j, Z(z)) = \sum_{j=1}^{l} \| v_j \otimes w_j - v_j \otimes P w_j \|^2$$

$$= \sum_{j=1}^{l} \| v_j \|^2 \| (\mathrm{Id}_{H_l} - P) w_j \|^2$$

$$= \operatorname{trace}(\mathrm{Id}_{H_l} - P).$$

Since the trace of $(\mathrm{Id}_{H_l} - P)$ is just the codimension in H_l of lin. span $\{z_1, \cdots, z_k\}$, and this codimension is at least $l - k$, the proof is complete.

PROOF OF PROPOSITION 2.1. Let

$$\Lambda = \{\lambda \in \hat{G}_k : m(\lambda) \geqq l\}.$$

Then a maximal spectral measure ν may be chosen with

$$\nu(\Lambda) = 1.$$

Denote by $L^2_{H_k \otimes H_l}(\Lambda, \nu)$ the Hilbert space of all square-integrable functions $y : \Lambda \to H_k \otimes H_l$. The inner product on this space is defined as follows:

$$\langle y_1, y_2 \rangle = \int_\Lambda \langle y_1(\lambda), y_2(\lambda) \rangle_{H_k \otimes H_l} d\nu(\lambda),$$

for all $y_1, y_2 \in L^2_{H_k \otimes H_l}(\Lambda, \nu)$.

Define a unitary representation Π'_g, $g \in G$, on $L^2_{H_k \otimes H_l}(\Lambda, \nu)$ as follows:

$$(\Pi'_g y)(\lambda) = (\lambda_g \otimes \mathrm{Id}_{H_l}) \cdot y(\lambda), \qquad \text{for all } y \in L^2_{H_k \otimes H_l}(\Lambda, \nu), \quad g \in G, \quad \lambda \in \Lambda.$$

Noting that Π' is just a version of $\int_\Lambda^\oplus l \cdot \lambda d\nu(\lambda)$, one sees from the definition of Λ that H contains a Π-invariant subspace H', such that the restriction of Π to H' is isomorphic to Π'.

Let Q denote the orthogonal projection from H onto H'. Then, for $y \in H'$, $z \in H$ and T a finite linear combination of the elements of $\{\Pi_g : g \in G\}$, one has

$$\begin{aligned} \| y - Tz \|^2 &\geqq \| Q(y - Tz) \|^2 \\ &= \| y - TQz \|^2 \\ &\geqq d^2(y, Z(Qz)). \end{aligned}$$

Taking the infimum over all possible finite linear combinations T, it follows that if $y \in H'$, then $d^2(y, Z(z)) \geqq d^2(y, Z(Qz))$, for all z in H.

Thus, it will be sufficient to show that there exist l orthonormal vectors $y_1, \cdots, y_l$ in H' with $\Sigma_{j=1}^l d^2(y_j, Z(z)) \geqq l - k$, for all z in H'. To prove this is equivalent to proving the proposition in the special case when $H = L^2_{H_k \otimes H_l}(\Lambda, \nu)$ and $\Pi = \Pi'$.

Letting $v_1, \cdots, v_l$ and $w_1, \cdots, w_l$ be as in Lemma 2.2, we choose as our orthonormal vectors in $L^2_{H_k \otimes H_l}(\Lambda, \nu)$ the constant functions $y_1, \cdots, y_l$ defined by:

$$y_j(\lambda) \equiv v_j \otimes w_j, \quad j = 1, \cdots, l, \quad \lambda \in \Lambda.$$

Now, let z be an arbitrary element of $L^2_{H_k \otimes H_l}(\Lambda, \nu)$, and denote by $P_{Z(z)}$ the orthogonal projection onto $Z(z)$. By a routine argument, it follows from the disjointness of the different representations $\lambda \otimes \mathrm{Id}_{H_l}$ as λ varies over Λ, that $P_{Z(z)}$ decomposes as follows:

For all $y \in L^2_{H_k \otimes H_l}(\Lambda, \nu)$, $(P_{Z(z)} \cdot y)(\lambda) = P_{Z(z(\lambda))} \cdot y(\lambda)$, ν-almost everywhere on Λ, where $P_{Z(z(\lambda))}$ denotes the orthogonal projection from $H_k \otimes H_l$ onto the closed $\lambda \otimes \mathrm{Id}_{H_l}$-cyclic subspace generated by $z(\lambda)$.

Hence, with the above choice of $y_1, \cdots, y_l$,

$$\sum_{j=1}^{l} d^2(y_j, Z(z)) = \sum_{j=1}^{l} \| y_j - P_{Z(z)} y_j \|^2$$

$$= \sum_{j=1}^{l} \left(\int_\Lambda \| y_j(\lambda) - P_{Z(z(\lambda))} y_j(\lambda) \|^2 d\nu(\lambda) \right)$$

$$= \int_\Lambda \sum_{j=1}^{l} d^2(v_j \otimes w_j, Z(z(\lambda))) d\nu(\lambda).$$

Since ν is a probability measure on Λ, and, by Lemma 2.2, the integrand in the above expression is greater than or equal to $l - k$ everywhere on Λ, one obtains

$$\sum_{j=1}^{l} d^2(y_j, Z(z)) \geqq l - k,$$

which is the desired result.

REMARK 2.3. The inequality given by Proposition 2.1 is precise in the sense that, for the given choice of $y_1, \cdots, y_l$, there always exists an element z of H with $\sum_{j=1}^{l} d^2(y_j, Z(z)) = \max(l-k, 0)$.

REMARK 2.4. If $m(\lambda) \leqq \dim \lambda$, ν-a.e. on $\hat{G}$, then the conclusion of the proposition is vacuous. This is all that could be expected, since this condition is equivalent to cyclicity of the representation Π.

§3. Approximations and spectral properties

The proof of the following theorem combines techniques of Chacon [2] and Stepin [13].

THEOREM 3.1. *Suppose that a measure (μ)-preserving Borel action, $(g, x) \to g \cdot x$, $g \in G$, $x \in X$, of a type I group, G, on a standard Borel space X admits a multiplicity N approximation with speed θ / n, $0 \leqq \theta < 2$. Then its spectral multiplicity function, m, satisfies the inequality*

$$m(\lambda) \leqq (2N/(2-\theta)) \cdot \dim \lambda,$$

for almost every λ in $\hat{G}$ (with respect to the maximal spectral type of the G-action).

PROOF. Fix a positive integer, n, and let

$$\xi(n) = \{C_{ij}(n) : i = 1, \cdots, q_j(n); j = 1, \cdots, N\}$$

and $g_{ij}(n)$, $i = 1, \cdots, q_j(n) - 1$, $j = 1, \cdots, N$, be as in Definition 1.3.

For each $j \in \{1, \cdots, N\}$, define group elements

$$h_{ij}(n) = \begin{cases} g_{i-1,j}(n) \cdot g_{i-2,j}(n) \cdots g_{1j}(n), & \text{for } i = 2, \cdots, q_j(n), \\ e \text{ (the group identity)}, & \text{for } i = 1, \end{cases}$$

and consider the subset of $C_{1j}(n)$ defined as follows:

$$A_j(n) = \bigcap_{i=1}^{q_j(n)} h_{ij}(n)^{-1} \cdot C_{ij}(n).$$

Clearly, $h_{ij}(n) \cdot A_j(n) \subset C_{ij}(n)$, for all $i = 1, \cdots, q_j(n)$. Furthermore, noting that whenever $x \in C_{1j}(n) \backslash A_j(n)$ there must exist a first $i \in \{2, \cdots, q_j(n)\}$ such that $x \notin h_{ij}(n)^{-1} \cdot C_{ij}(n)$, we have, for each $i = 1, \cdots, q_j(n)$,

$$\begin{aligned} \mu(C_{ij}(n) \backslash h_{ij}(n) \cdot A_j(n)) &= \mu(C_{1j}(n)) - \mu(A_j(n)) \\ &\leqq \sum_{i=1}^{q_j(n)=1} \mu(h_{ij}(n)^{-1} \cdot C_{ij}(n) \backslash h_{i+1,j}(n)^{-1} \cdot C_{i+1,j}(n)) \\ &= \frac{1}{2} \sum_{i=1}^{q_j(n)-1} \mu(g_{ij}(n) \cdot C_{ij}(n) \Delta C_{i+1,j}(n)) \qquad (1) \\ &\leqq (\tfrac{1}{2}) \cdot (\theta / q_j(n)) \cdot (q_j(n) \mu(C_{1j}(n))) \qquad \text{(by Definition 1.3)} \\ &= (\theta/2) \cdot \mu(C_{ij}(n)). \end{aligned}$$

This inequality will be used to show that, with respect to the unitary representation Π induced from the given group action, one has

$$(2) \quad \limsup_{n \to \infty} \sum_{j=1}^{N} d^2(y, Z(\chi_{A_j(n)})) \leqq (N - 1 + (\theta/2)) \|y\|^2, \quad \text{for all } y \in L^2(X, \mu),$$

where, in accordance with the notation of §2, for each j, $Z(\chi_{A_j(n)})$ denotes the Π-cyclic subspace of $L^2(X, \mu)$ generated by the characteristic function of the set $A_j(n)$, and d denotes the distance induced from the norm on $L^2(X, \mu)$.

To obtain (2), fix y, and, for each n, let

$$y(n) = \sum_{j=1}^{N} \sum_{i=1}^{q_j(n)} a_{ij}(n) \chi_{C_{ij}(n)}$$

be the projection of y onto the subspace of $L^2(X,\mu)$ spanned by $\{\chi_{C_{ij}(n)}: i=1,\cdots,q_j(n);\ j=1,\cdots,N\}$. Then

$$\begin{aligned}
d^2(y(n),Z(\chi_{A_j(n)})) &\leqq \left\| y(n)-\left(\sum_{i=1}^{q_j(n)} a_{ij}(n)\cdot\Pi_{h_{ij}(n)}\chi_{A_j(n)}\right)\right\|^2\\
&= \left\| y(n)-\left(\sum_{i=1}^{q_j(n)} a_{ij}(n)\cdot\chi_{h_{ij}(n)\cdot A_j(n)}\right)\right\|^2\\
&= \sum_{\substack{k=1\\k\neq j}}^{N}\left(\sum_{i=1}^{q_k(n)} |a_{ik}(n)|^2\mu(C_{ik}(n))\right)\\
&\quad + \sum_{i=1}^{q_j(n)} |a_{ij}(n)|^2\mu(C_{ij}(n)\backslash h_{ij}(n)\cdot A_j(n))\\
&\leqq \sum_{\substack{k=1\\k\neq j}}^{N}\left(\sum_{i=1}^{q_k(n)} |a_{ik}(n)|^2\mu(C_{ik}(n))\right)\\
&\quad + (\theta/2)\cdot\sum_{i=1}^{q_j(n)} |a_{ij}(n)|^2\mu(C_{ij}(n)) \qquad \text{(by (1))}.
\end{aligned}$$

Summing over j, this gives

$$\sum_{j=1}^{N} d^2(y(n),Z(\chi_{A_j(n)}))\leqq (N-1+(\theta/2))\|y(n)\|^2.$$

Inequality (2) now follows, because the hypothesis that $\xi(n)\rightarrow\varepsilon$ implies that $y(n)\rightarrow y$ as $n\rightarrow\infty$.

We are now in a position to use Proposition 2.1. Fix a positive integer k. Let us denote by m_k the essential supremum of the restriction to $\hat{G}_k$ of the spectral multiplicity function m. Choose an arbitrary positive integer l no greater than m_k. Then, by 2.1, there exist unit vectors $y_1,\cdots,y_l$ in $L^2(X,\mu)$ such that

$$\sum_{s=1}^{l} d^2(y_s,Z(z))\geqq l-k, \quad \text{for all } z\in L^2(X,\mu).$$

Applying this inequality with $z=\chi_{A_j(n)}$, and summing over j, gives

$$\sum_{s=1}^{l}\sum_{j=1}^{N} d^2(y_s,Z(\chi_{A_j(n)}))\geqq N(l-k).$$

Inequality (2) now implies that

$$l(N-1+(\theta/2))\geqq N(l-k).$$

When θ is less than 2, this is equivalent to:

$$l \leqq 2Nk/(2-\theta).$$

Since l was defined to be an arbitrary positive integer less than or equal to m_k, we conclude that

$$m_k \leqq 2Nk/(2-\theta), \quad \text{for all } k \in \{1,2,\cdots\}.$$

This completes the proof.

REMARK 3.2. When G is abelian, so that $\dim \lambda = 1$ for all λ in $\hat{G}$, Theorem 3.1 gives the uniform bound $[2N/(2-\theta)]$ on the spectral multiplicity of the G-action. If θ is less than $2/(N+1)$, this bound is equal to N, the best that could be expected using multiplicity N approximations.

Chacon [2] obtained the bound N for the special case (see Remark 1.5) of a multiplicity N approximation, with speed θ/n, $0 \leqq \theta < 2/(N+1)$, of a measure-preserving automorphism of probability space. Stepin [13] showed that a measure-preserving automorphism of a probability space which admits a cyclic approximation with speed θ/n, $\theta < 2$, must have spectral multiplicity uniformly less than or equal to $[2/(2-\theta)]$.

REMARK 3.3. Returning to the general case of an action of a type I group, observe that if one is given a multiplicity N approximation with speed $o(1/n)$, then the theorem implies that

$$m(\lambda) \leqq N \cdot \dim \lambda, \quad \text{for almost all } \hat{G}.$$

This inequality may be interpreted as bounding by N the number of cyclic components needed to make up the unitary representation induced from the group action (see Remark 2.4).

The next two theorems are straightforward generalizations of theorem 1 of [13] and theorem 2.1 of [6]. The reader may check that the proofs of these earlier results can be extended so as to cover our more general situation.

THEOREM 3.4 [13]. *Let there be given a measure (μ)-preserving action of a non-compact, locally compact, second countable, abelian group G on a standard Borel space X. Suppose that there exists a sequence of semi-partitions $\xi(n)$, $n = 1,2,\cdots$, and a sequence of group elements $g(1)$, $g(2),\cdots$ satisfying*

(i) $\xi(n) \to \varepsilon$, *as* $n \to \infty$,

(ii) $\lim_{n\to\infty} g(n) = \infty$, *and*

(iii) *there exists a constant* $\theta < 1$, *independent of* n, *such that, whenever* $C \in \xi(n)$,

$$\mu(g(n)\cdot C\Delta C) \leqq \theta \cdot \mu(C).$$

Then the maximal spectral type of the group action is singular with respect to the Haar measure on the dual group $\hat{G}$.

THEOREM 3.5 [6]. *Suppose that a measure-preserving group action admits a cyclic approximation with speed* θ/n, $\theta \geqq 0$.

Then that action has a finite ergodic decomposition. If $\theta < 4$, *then the action is ergodic. In general, the number of ergodic components is no more than* $\max(1, \theta/2)$.

REMARK 3.6. Unlike the other results of this paper, Theorem 3.5 applies to actions of arbitrary groups.

REMARK 3.7. The hypotheses of Theorem 3.4 are satisfied if the group action admits a cyclic approximation with speed θ/n, $0 \leqq \theta < 1$, provided that the group elements $g_1(n), \cdots, g_{q(n)}(n)$, $n = 1, 2, \cdots$, specified in Definition 1.4 may be chosen so that

$$\lim_{n\to\infty} g_{q(n)} \cdot g_{q(n)-1}(n) \cdots g_1(n) = \infty.$$

§4. Applications

As an illustration of the general techniques developed in the previous sections, we have the following elementary example.

EXAMPLE 4.1. Let γ_1 and γ_2 be rationally independent real numbers. Consider the $\mathbf{Z}^2$-action on the real line defined as follows:

$$(k_1, k_2)\cdot x = x + k_1\gamma_1 + k_2\gamma_2, \quad \text{for all } (k_1, k_2) \in \mathbf{Z}^2, \quad \text{and } x \in \mathbf{R}.$$

This action preserves the Lebesgue measure, denoted μ.

The rational independence of γ_1 and γ_2 implies that, if C and C' are arbitrary sub-intervals of $\mathbf{R}$ of the same finite length, then, given any $\varepsilon > 0$, arbitrarily large integers k_1 and k_2 may be found so that $\mu((k_1, k_2)\cdot C\Delta C')$ is less than ε. From this it is clear that the sequence of semi-partitions

$$\xi(n) = \{[i/n, i + 1/n) : i = -n^2, -n^2 + 1, \cdots, n^2\}, \quad n = 1, 2, \cdots$$

provides cyclic approximations of the given $\mathbf{Z}^2$-action with arbitrary speed. This sequence of semi-partitions also satisfies the conditions of Theorem 3.4. Hence, from the results of the previous section one may deduce that the given $\mathbf{Z}^2$-action is ergodic and has simple, singular spectrum. (Note that it is possible to obtain the spectral decomposition of this $\mathbf{Z}^2$-action by direct means.)

We now proceed to a study of the class of cylinder transformations, $\{T_{\alpha,\beta} : \alpha, \beta \in (0,1)\}$, defined in the introduction. The space $[0,1) \times \mathbf{R}$, on which each $T_{\alpha,\beta}$ acts, is taken to be equipped with the product, denoted μ, of the Lebesgue measures on $[0,1)$ and $\mathbf{R}$ respectively.

For each $x \in \mathbf{R}$,

$$[x] \text{ denotes the integer part of } x,$$

$$\langle x \rangle = x - [x], \quad \text{the fractional part of } x,$$

and

$$\langle\langle x \rangle\rangle = \min(\langle x \rangle, 1 - \langle x \rangle), \text{ the distance from } x \text{ to the nearest integer.}$$

If l is a positive integer, then $T^l_{\alpha,\beta}$ acts on a point (x, t) in $[0,1) \times \mathbf{R}$ as follows:

$$T^l_{\alpha,\beta}(x,t) = \left(\langle x + l\alpha \rangle, t + \sum_{i=0}^{l-1} \chi_{[0,\beta)}(\langle x + i\alpha \rangle) - l\beta \right).$$

In order to estimate the vertical components of the translations brought about by iterates of $T_{\alpha,\beta}$, we introduce the sequence of "discrepancies"

$$D_l(\alpha) = \sup_{0 \leq a < b \leq 1} \left| (1/l) \sum_{i=0}^{l-1} \chi_{[a,b)}(\langle i\alpha \rangle) - (b - a) \right|, \qquad l = 1, 2, \cdots.$$

Note that, for each positive integer l, the vertical distance through which $T^l_{\alpha,\beta}$ shifts a point in $[0,1) \times \mathbf{R}$ is never more than $2lD_l(\alpha)$. The sequences of discrepancies, $D_l(\alpha)$, $l = 1, 2, \cdots$, α irrational, have long been studied in connection with the irregularity of distribution of the sequences $i\alpha$, $i = 1, \cdots$ (see Kuipers and Niederreiter [9]). We shall have use of the well known result that, for any irrational α, the sequence $lD_l(\alpha)$, $l = 1, 2, \cdots$, is unbounded.

PROPOSITION 4.2. *Suppose that α is an element of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n = 1, 2, 3, \cdots$, such that, as $n \to \infty$,*

(i) $q_n \nearrow \infty$,

(ii) $s_n q_n^2 |\alpha - p_n/q_n| \to 0$, *where* $s_n = \sup_{1 \le l \le q_n} lD_l(\alpha)$ *for each* n.
Then, whenever $\beta \in (0,1)$ *satisfies*

(iii) $\liminf_{n\to\infty} \max(\langle\langle q_n\beta\rangle\rangle, (1/\langle\langle q_n\beta\rangle\rangle)\cdot s_n q_n^2|\alpha - p_n/q_n|) = 0$,
the transformation $T_{\alpha,\beta}$ *has spectral multiplicity uniformly equal to one.*

Proof. Note that in order that condition (iii) be satisfied, β must be irrational. Hence $\langle\langle q_n\beta\rangle\rangle$ is non-zero for all n. Also, by going to a subsequence of the given sequence of irreducible fractions, we may assume that β satisfies

$$\text{(iii)}' \qquad \begin{aligned} &\lim_{n\to\infty} \langle\langle q_n\beta\rangle\rangle = 0, \\ &\lim_{n\to\infty} (s_n/\langle\langle q_n\beta\rangle\rangle)\cdot q_n^2|\alpha - p_n/q_n| = 0. \end{aligned}$$

We shall show that, under these conditions, the ($\mathbf{Z}$-action generated by the) transformation $T_{\alpha,\beta}$ admits a multiplicity one approximation with speed $o(1/n)$.

Fix a positive integer n, and split the space $[0,1)\times\mathbf{R}$ into a disjoint union of the $2q_n$ "columns" of the two "types" $E_k(n)$ and $F_k(n)$, $k = 0,\cdots,q_n - 1$, defined by

$$E_k(n) = \left[k/q_n, \frac{k + \langle q_n\beta\rangle}{q_n}\right)\times\mathbf{R}$$

and

$$F_k(n) = \left[\frac{k + \langle q_n\beta\rangle}{q_n}, \frac{k+1}{q_n}\right)\times\mathbf{R}.$$

The effect of the transformation $T_{p_n/q_n,\beta}$ on any of these columns is a rigid translation with horizontal component p_n/q_n (mod 1) and vertical component either $1-\beta$ or $-\beta$, according to whether the column in question lies to the left or right, respectively, of the vertical line

$$L = \{(x,t)\in[0,1)\times\mathbf{R} : x = \beta\}.$$

Note that each column lies entirely to one side or other of L. Focussing on the horizontal component of this translation, we see that for each $k \in \{0,\cdots,q_n - 1\}$,

$$T^k_{p_n/q_n,\beta}E_k(n) = E_{k+p_n(\mathrm{mod}\, q_n)}(n)$$

and

$$T^k_{p_n/q_n,\beta}F_k(n) = F_{k+p_n(\mathrm{mod}\, q_n)}(n).$$

Thus, the irreducibility of the fraction p_n/q_n implies that, as a permutation of either $\{E_0(n),\cdots,E_{q_n-1}(n)\}$ or $\{F_0(n),\cdots,F_{q_n-1}(n)\}$, $T_{p_n/q_n,\beta}$ is cyclic. Hence, to determine the net translation of a column under q_n iterations of $T_{p_n/q_n,\beta}$, it is necessary only to count the number of columns of the same type lying on either side of L. The columns to the left of L are $E_0(n),\cdots,E_{[q_n\beta]}$ and $F_0(n),\cdots,F_{[q_n\beta]-1}(n)$. Upon making the calculations

$$([q_n\beta]+1)(1-\beta)-(q_n-([q_n\beta]+1))\beta = 1-\langle q_n\beta\rangle$$

and

$$[q_n\beta](1-\beta)-(q_n-[q_n\beta])\beta = -\langle q_n\beta\rangle,$$

we conclude that $T^{q_n}_{p_n/q_n,\beta}$ translates each $E_k(n)$, $k=0,\cdots,q_n-1$, vertically upwards by $1-\langle q_n\beta\rangle$ units, whereas each $F_k(n)$, $k=0,\cdots,q_n-1$, is translated vertically downwards by $\langle q_n\beta\rangle$ units.

The above observations allow us to use $T_{p_n/q_n,\beta}$ to define our n-th approximating semi-partition,

$$\xi(n)=\{C_i(n): i=1,\cdots,6r_nq_n\},$$

by choosing

$$C_i(n)=T^{i-1}_{p_n/q_n,\beta}C_1(n), \qquad \text{for each } i\in\{1,\cdots,6r_nq_n\},$$

with $r_n=[(s_n+1)/\langle\langle q_n\beta\rangle\rangle]$ and

$$C_1(n)=\begin{cases}\{(x,t)\in E_0(n): -3r_n\langle\langle q_n\beta\rangle\rangle \leqq t<(-3r_n+1)\langle\langle q_n\beta\rangle\rangle\}, \\ \qquad\qquad\qquad\qquad \text{if } \langle\langle q_n\beta\rangle\rangle = 1-\langle q_n\beta\rangle, \\ \{(x,t)\in F_0(n): (3r_n-1)\langle\langle q_n\beta\rangle\rangle \leqq t<3r_n\langle\langle q_n\beta\rangle\rangle\}, \\ \qquad\qquad\qquad\qquad \text{if } \langle\langle q_n\beta\rangle\rangle = \langle q_n\beta\rangle.\end{cases}$$

Each element of $\xi(n)$ is a rectangle of base $(1-\langle q_n\beta\rangle)/q_n$, filling a horizontal strip of depth $\langle\langle q_n\beta\rangle\rangle$ across the column in which it lies. In the case when $\langle\langle q_n\beta\rangle\rangle = 1-\langle q_n\beta\rangle$ (respectively $\langle q_n\beta\rangle$), the rectangles $C_1(n),\cdots,C_{q_n}(n)$ lie, one in each of the columns $E_0(n),\cdots,E_{q_n-1}(n)$ (respectively $F_0(n),\cdots,F_{q_n-1}(n)$), forming a pattern in which no element is vertically displaced with respect to $C_1(n)$ by more than $2s_n$ units (see the remarks accompanying the definition of the sequence of discrepancies). The other elements of $\xi(n)$ may be obtained by taking $6r_n-1$ successive vertical shifts of this basic pattern, upward (respectively downward) by $\langle\langle q_n\beta\rangle\rangle$ units. This implies that the elements of $\xi(n)$ are indeed disjoint, and by the choice of r_n, that

$$(*)\qquad \bigcup_{C\in\xi(n)} C \supset \begin{cases} \left(\bigcup_{k=0}^{q_n-1} E_k(n)\right)\cap([0,1)\times[-s_n,s_n)), & \text{if } \langle\langle q_n\beta\rangle\rangle = 1-\langle q_n\beta\rangle,\\ \left(\bigcup_{k=0}^{q_n-1} F_k(n)\right)\cap([0,1)\times[-s_n,s_n)), & \text{if } \langle\langle q_n\beta\rangle\rangle = \langle q_n\beta\rangle. \end{cases}$$

Note that, for all Borel subsets A of $[0,1)\times\mathbf{R}$ with $\mu(A)<\infty$, the following hold:

$$\lim_{\langle q_n\beta\rangle\to 1}\mu\left(\left(\bigcup_{k=0}^{q_n-1} E_k(n)\right)\cap A\right)=\mu(A)$$

and

$$\lim_{\langle q_n\beta\rangle\to 0}\mu\left(\left(\bigcup_{k=0}^{q_n-1} F_k(n)\right)\cap A\right)=\mu(A).$$

Since $s_n\to\infty$, and the dimensions of the rectangles in $\xi(n)$ tend uniformly to zero as $n\to\infty$, we can thus conclude from (*) that $\xi(n)\to\varepsilon$ as $n\to\infty$.

Furthermore,

$$\begin{aligned}
&(1/(6r_nq_n\mu(C_1)))\cdot\sum_{i=1}^{6r_nq_n-1}\mu(T_{\alpha,\beta}C_i(n)\Delta C_{i+1}(n))\\
&=\frac{(6r_nq_n-1)\cdot 2|\alpha-p_n/q_n|\langle\langle q_n\beta\rangle\rangle}{6r_nq_n\cdot((1-\langle\langle q_n\beta\rangle\rangle)/q_n)\langle\langle q_n\beta\rangle\rangle}\\
&=(1/6r_nq_n)\cdot\frac{(12r_nq_n-2)q_n|\alpha-p_n/q_n|}{(1-\langle\langle q_n\beta\rangle\rangle)}\\
&=o(1/6r_nq_n),
\end{aligned}$$

by (iii)′ and the choice of r_n, $n=1,2,\cdots$.

Hence, the semi-partitions $\xi(n)$, $n=1,2,\cdots$, define a multiplicity one approximation of $T_{\alpha,\beta}$ with speed $o(1/n)$. It only remains to apply Theorem 3.1.

Under the conditions of Proposition 4.2, the transformation $T_{\alpha,\beta}$ must have singular spectral type. This follows from:

PROPOSITION 4.3. *Suppose that α and β are irrational elements of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n=1,2,\cdots$, with*

(i) $q_n\nearrow\infty$, *as* $n\to\infty$,

(ii) *for all* n, $2q_n^2|\alpha - p_n/q_n| \leqq \theta < 1$, *where* θ *is some constant, independent of* n,

(iii) $\liminf_{n\to\infty} \langle\langle q_n\beta\rangle\rangle = 0$.

Then the cylinder transformation $T_{\alpha,\beta}$ *has singular spectral type.*

PROOF. Choose a positive constant, a, with $\theta + 2/a < 1$. By going to a subsequence of the given sequence of irreducible fractions, we may assume that

$$\langle\langle q_n\beta\rangle\rangle \to 0 \qquad \text{as } n \to \infty,$$

and that there exists a constant $\theta' < 1$ with

$$\theta + (2/a) + 2\langle\langle q_n\beta\rangle\rangle \leqq \theta', \qquad \text{for each } n. \tag{1}$$

Consider an arbitrary rectangle in $[0,1)\times\mathbf{R}$ of the form

$$C = [k/q_n, (k+1)/q_n) \times [t, t + a\langle\langle q_n\beta\rangle\rangle), \quad \text{with } k \in \{0,\cdots,q_n - 1\}, \quad t \in \mathbf{R}.$$

Let C' be the subrectangle defined by:

$$C' = \begin{cases} [k/q_n, (k + \langle q_n\beta\rangle)/q_n) \times [t, t + a\langle\langle q_n\beta\rangle\rangle), \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = 1 - \langle q_n\beta\rangle, \\ \\ [(k + \langle q_n\beta\rangle)/q_n, (k+1)/q_n) \times [t, t + a\langle\langle q_n\beta\rangle\rangle), \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = \langle q_n\beta\rangle. \end{cases}$$

Note that

$$\mu(C\Delta C') = \langle\langle q_n\beta\rangle\rangle\,\mu(C). \tag{2}$$

From the proof of Proposition 4.2, it is clear the $T^{q_n}_{p_n/q_n,\beta}C'$ is just a vertical translate of C' by $\pm\langle\langle q_n\beta\rangle\rangle$ units. Hence,

$$\mu(T^{q_n}_{p_n/q_n,\beta}C'\Delta C') = (2/a)\mu(C'). \tag{3}$$

Now,

$$\begin{aligned} \mu(T^{q_n}_{\alpha,\beta}C'\Delta T^{q_n}_{p_n/q_n,\beta}C') &\leqq \sum_{i=0}^{q_n-1} \mu(T^{q_n-i}_{\alpha,\beta}T^{i}_{p_n/q_n,\beta}C'\Delta T^{q_n-i-1}_{\alpha,\beta}T^{i-1}_{p_n/q_n,\beta}C') \\ &= \sum_{i=0}^{q_n-1} \mu(T_{\alpha,\beta}\cdot T^{i}_{p_n/q_n,\beta}C'\Delta T_{p_n/q_n,\beta}\cdot T^{i}_{p_n/q_n,\beta}C') \\ &= q_n \cdot 2|\alpha - p_n/q_n|\,a\langle\langle q_n\beta\rangle\rangle \\ &\leqq \theta\mu(C), \qquad \text{by condition (ii).} \end{aligned} \tag{4}$$

Here, we have used the fact that under any iterate of $T_{p_n/q_n,\beta}$, the rectangle C' is translated to another rectangle of the same type. This is apparent from the proof of Proposition 4.2.

Together, (1), (2), (3) and (4) imply that, for any rectangle C of the assumed type,

$$\mu(T^{q_n}_{\alpha,\beta}C\Delta C)\leqq \theta'\mu(C).$$

Since β is irrational, $\langle\langle q_n\beta\rangle\rangle$ is never zero, and the proof of the proposition may be completed by applying Theorem 3.4 to the sequence of partitions defined as follows:

$$\xi(n)=\{[k/q_n,(k+1)/q_n)\times[la\langle\langle q_n\beta\rangle\rangle,(l+1)a\langle\langle q_n\beta\rangle\rangle):k=0,\cdots,q_n-1,\ l\in\mathbf{Z}\},$$
$$\text{for all } n=1,2,\cdots.$$

Now, consider the case when the parameter β is rational, say equal to c/d in lowest terms. Then, for all α in $(0,1)$, any vertical translate of $[0,1)\times d^{-1}\cdot\mathbf{Z}$ is a $T_{\alpha,\beta}$-invariant subset of $[0,1)\times\mathbf{R}$. Thus, we are led to define a new measure-preserving automorphism, denoted $S_{\alpha,c/d}$, by taking the restriction of $T_{\alpha,c/d}$ to the space $[0,1)\times d^{-1}\cdot\mathbf{Z}$ (equipped with the obvious product of Lebesgue and counting measures). The class of transformations $\{S_{\alpha,c/d}:\alpha,c/d\in(0,1)\}$ is of interest in that it provides examples of approximations of all finite multiplicities:

Proposition 4.4. *Let c/d be an irreducible fraction in $(0,1)$. Suppose that $\alpha\in(0,1)$ is such that there exists a sequence of irreducible fractions p_n/q_n, $n=1,2,\cdots$ satisfying*

(i) *no q_n is a multiple of d,*

(ii) *$q_n\nearrow\infty$ as $n\to\infty$,*

and

(iii) *$s_nq_n^2|\alpha-p_n/q_n|\to 0$, as $n\to\infty$, where, as before,*

$$s_n=\sup_{1\leqq l\leqq q_n} lD_l(\alpha),\quad \textit{for each } n.$$

Then the transformation $S_{\alpha,c/d}$ admits a multiplicity d approximation with speed $o(1/n)$ and, hence, has spectral multiplicity less than or equal to d.

Proof. Put $b_n=d\langle q_nc/d\rangle$. Then, by (i), b_n belongs to $\{1,2,\cdots,d-1\}$, for each $n=1,2,\cdots$.

From the proof of Proposition 4.2 it is apparent that the columns, defined this time by

$$E_k(n) = [k/q_n, k/q_n + b_n/dq_n) \times d^{-1}\cdot \mathbf{Z}$$

and

$$F_k(n) = [k/q_n + b_n/dq_n, (k+1)/q_n) \times d^{-1}\cdot \mathbf{Z},$$

for $k \in \{0, \cdots, q_n - 1\}$, have the following properties with respect to the restricted transformation $S_{p_n/q_n, c/d}$:

(a) each column is rigidly translated by $S_{p_n/q_n, c/d}$ onto another column of the same type,

(b) as a permutation of either $\{E_0(n), \cdots, E_{q_n-1}(n)\}$ or $\{F_0(n), \cdots, F_{q_n-1}(n)\}$, $S_{p_n/q_n, c/d}$ is cyclic,

(c) under $S^{q_n}_{p_n/q_n, c/d}$, each of the columns $E_0(n), \cdots, E_{q_n-1}(n)$ moves vertically upwards by $1 - \langle q_n c/d\rangle$ units ($d - b_n$ "levels"), and

(d) under $S^{q_n}_{p_n/q_n, c/d}$, each of the columns $F_0(n), \cdots, F_{q_n-1}(n)$ is translated vertically downwards by $\langle q_n c/d\rangle$ units (b_n "levels").

Note that by a level of a column we mean one of the doubly infinite stack of intervals of which it is composed.

These properties lead us to choose our sequence of approximating semipartitions as follows:

For each $n = 1, 2, \cdots$, we put

$$\xi(n) = \{C_{ij}(n) : i = 1, \cdots, 6r_n q_n;\ j = 1, \cdots, d\},$$

where we define

$$C_{ij}(n) = S^{i-1}_{p_n/q_n, c/d} C_{1j}(n), \qquad \text{for all } i \text{ and } j,$$

with

$$r_n = d[s_n + 1]$$

and

$$C_{1j}(n) = \begin{cases} [0, b_n/dq_n) \times \{(-3r_n + j)/d\}, & \text{for } j = 1, \cdots, d - b_n, \\ [b_n/dq_n, 1/q_n) \times \{(3r_n + j - d)/d\}, & \text{for } j = d - b_n + 1, \cdots, d. \end{cases}$$

The intervals $C_{11}(n), \cdots, C_{1,d-b_n}(n)$ (respectively $C_{1,d-b_n+1}(n), \cdots, C_{1d}(n)$) have been chosen to be successive levels of the column $E_0(n)$ (respectively $F_0(n)$). This implies, using (a) and (b) above, that $\xi(n)$ admits the following subdivision by columns:

For each $k \in \{1, \cdots, q_n\}$, whenever $i \in \{k + lq_n : l = 0, \cdots, 6r_n - 1\}$, the elements $C_{i,1}(n),\cdots,C_{i,d-b_n}(n)$ (respectively $C_{i,d-b_n+1}(n),\cdots,C_{i,d}(n)$) are successive levels of the column $S^{k-1}_{p_n/q_n,c/d}E_0(n)$ (respectively $S^{k-1}_{p_n/q_n,c/d}F_0(n)$). Note that each of the columns $E_0(n),\cdots,E_{q_n-1}(n)$ (respectively $F_0(n),\cdots,F_{q_n-1}(n)$) is of the form $S^{k-1}_{p_n/q_n,c/d}E_0(n)$ (respectively $S^{k-1}_{p_n/q_n,c/d}F_0(n)$), for some $k\in\{1,\cdots,q_n\}$.

Now, for fixed $k\in\{1,\cdots,q_n\}$, property (c) (respectively (d)), above, implies that if $i\in\{k+lq_n : l=0,\cdots,6r_n-2\}$, then the intervals $C_{i+q_n,1}(n)$, $C_{i+q_n,2}(n)$, $\cdots$, $C_{i+q_n,d-b_n}(n)$ (respectively $C_{i+q_n,d-b_n+1}(n)$, $\cdots$, $C_{i+q_n,d}(n)$) occupy the next $(d-b_n)$-tuple of successive levels of $S^{k-1}_{p_n/q_n,c/d}E_0(n)$ above (respectively the next b-tuple of successive levels of $S^{k-1}_{p_n/q_n,c/d}F_0(n)$ below) that occupied by $C_{i,1}(n),\cdots,C_{i,d-b_n}(n)$ (respectively $C_{i,d-b_n+1}(n)$, $\cdots$, $C_{i,d}(n)$). Thus, within the columns $E_0(n),\cdots,E_{q_n-1}(n)$ (respectively $F_0(n),\cdots,F_{q_n-1}(n)$), the semi-partition $\xi(n)$ contains stacks of $6r_n(d-b_n)$ (respectively $6r_nb_n$) consecutive levels. The lowermost (respectively uppermost) levels in these stacks are, modulo a reordering, $C_{1,1}(n)$, $C_{2,1}(n)$, $\cdots$, $C_{q_n,1}(n)$ (respectively $C_{1,d}(n)$, $C_{2,d}(n)$, $\cdots$, $C_{q_n,d}(n)$), none of which, by the definition of s_n, is more than $2s_n$ units vertically displaced from the $3r_n$th level below (respectively above) the horizontal axis. Since, by the definition of r_n, a vertical displacement of $2s_n$ units spans no more than $2r_n$ levels, we see that

$$\{(x,j/d): x\in[0,1),\ j\in\{-r_n,-r_n+1,\cdots,r_n\}\}\subset\bigcup_{C\in\xi(n)}C,\qquad\text{for all } n.$$

Together with the uniform convergence to zero of the lengths of the elements of $\xi(n)$, this inclusion implies that $\xi(n)\to\varepsilon$ as $n\to\infty$.

The rest of the proof is exactly as for 4.2. The details are left to the reader.

The following proposition may be proved in the same way as Proposition 4.3.

PROPOSITION 4.5. *Let α be an element of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n=1,2,\cdots$, satisfying*

(i) *each q_n is a multiple of d,*

(ii) *$q_n\nearrow\infty$, as $n\to\infty$,*

(iii) *there exists a constant $\theta<1$ with*

$$2q_n^2|\alpha-p_n/q_n|\leqq\theta,\qquad\text{for all } n.$$

Then the transformation $S_{\alpha,c/d}$ has singular spectral type.

REMARK 4.6. The conditions on the parameter α hypothesized in Propositions 4.2–4.5, respectively, are satisfied on residual subsets of the parameter space $(0,1)$. In the cases of 4.3 and 4.5, these subsets have Lebesgue measure one.

It may be shown that, for any α which satisfies the conditions of either 4.2 or 4.3, the corresponding condition on the parameter β is satisfied by a residual, measure-one subset of values in $(0,1)$.

REMARK 4.7. From the proofs of Propositions 4.2 and 4.4, the transformations $T_{p_n/q_n,\beta}$ and $S_{p_n/q_n,c/d}$ used in approximating $T_{\alpha,\beta}$ and $S_{\alpha,c/d}$, respectively, may be seen to have infinite Lebesgue spectrum (in fact, these transformations are dissipative — each have a wandering set whose transforms cover the whole space). The question arises whether either of the transformations $T_{\alpha,\beta}$ or $S_{\alpha,c/d}$ may have Lebesgue spectral type (with finite multiplicity?) for some irrational value of α?

REMARK 4.8. It follows from [11] that the conditions of Proposition 4.4 are sufficient for ergodicity of $S_{\alpha,c/d}$.

The following proposition summarizes a number of spectral observations which do not depend upon the method of approximations. Note that the maximal spectral type and spectral multiplicity functions of any measure-preserving transformation are defined on the circle group $K = \{z \in \mathbf{C} : |z| = 1\}$.

PROPOSITION 4.9. *Let α and β be arbitrary elements of* $(0,1)$. *Then*

(a) *the transformation $T_{\alpha,\beta}$ has maximal spectral type and spectral multiplicity function both invariant under each of the following transformations of the circle group*:

(i) $z \to \bar{z}$, $z \in K$, *the reflection in the horizontal axis,*

(ii) $z \to e^{2\pi i\alpha} \cdot z$, $z \in K$, *the rotation through the angle* $2\pi\alpha$,

(iii) $z \to e^{2\pi i\beta} \cdot z$, $z \in K$, *the rotation through the angle* $2\pi\beta$;

(b) *when β is rational, the same is true of the maximal spectral type and spectral multiplicity function of the transformation $S_{\alpha,\beta}$*;

(c) *in the special case when β equals $\frac{1}{2}$, the spectral multiplicity function of $S_{\alpha,\beta}$ is even almost everywhere on K.*

PROOF. For each $t \in \mathbf{R}$, let $V_{\alpha,\beta,t}$ denote the unitary operator defined as follows on $L^2[0,1)$:

$$(V_{\alpha,\beta,t}y)(x) = \exp(-2\pi it(\chi_{[0,\beta)}(x) - \beta)) \cdot y(x + \alpha \pmod 1),$$

$$\text{for all } y \in L^2[0,1) \text{ and } x \in [0,1).$$

By conjugating with the Fourier transform in the **R**-coordinate, it is not difficult to show that the unitary operator induced on $L^2([0,1)\times\mathbf{R})$ by $T_{\alpha,\beta}$ is unitarily equivalent to $\int_{\mathbf{R}}^{\oplus} V_{\alpha,\beta,t}dt$, the direct integral operator on the space $\int_{\mathbf{R}}^{\oplus} L^2[0,1)dt$ which acts on each norm-square-integrable vector field $t\to y_t:\mathbf{R}\to L^2[0,1)$ as follows:

$$\left(\left(\int_{\mathbf{R}}^{\oplus} V_{\alpha,\beta,t}dt\right)\cdot y\right)_t = V_{\alpha,\beta,t}y_t, \qquad \text{for all } t\in\mathbf{R}.$$

Similarly, if c/d is an irreducible fraction in $(0,1)$, then the unitary operator induced on $L^2([0,1)\times d^{-1}\mathbf{Z})$ by $S_{\alpha,c/d}$ may be shown to be unitarily equivalent to $\int_{[0,d)}^{\oplus} V_{\alpha,c/d,t}dt$.

Now, check that the identities

(i) $W_1^* V_{\alpha,\beta,t} W_1 = V_{\alpha,\beta,-t}^* \quad (= V_{\alpha,c/d,d-t}^*,$ if $\beta = c/d)$,

(ii) $W_2^* V_{\alpha,\beta,t} W_2 = e^{2\pi i\alpha} V_{\alpha,\beta,t}$, and

(iii) $V_{\alpha,\beta,t-1} = e^{2\pi i\beta} V_{\alpha,\beta,t}$

hold for all possible values of the parameters, where W_1 and W_2 are the unitary operators on $L^2[0,1)$ defined by setting

$$W_1 y(x) = y(\alpha+\beta-x \pmod 1)$$

and

$$W_2 y(x) = e^{2\pi ix}y(x), \qquad \text{for all } y\in L^2[0,1) \text{ and } x\in[0,1).$$

The identities (i), (ii), and (iii), applied to the stated direct integral decompositions of the unitary operators induced from $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$, imply that each of these operators is unitarily equivalent to (i) its adjoint, (ii) itself multiplied by $e^{2\pi i\alpha}$, and (iii) itself multiplied by $e^{2\pi i\beta}$, respectively. These unitary equivalences prove parts (a) and (b) of the statement of the proposition.

Now, note that

$$W_3^* V_{\alpha,\frac12,t} W_3 = V_{\alpha,\frac12,2-t}, \qquad \text{for all } \alpha\in(0,1), \quad t\in\mathbf{R},$$

where W_3 is the unitary operator defined on $L^2[0,1)$ as follows:

$$W_3 y(x) = y(x+\tfrac12 \pmod 1)), \qquad \text{for all } y\in L^2[0,1) \text{ and } x\in[0,1).$$

It follows from this identity that the direct integral decomposition of the unitary operator induced from $S_{\alpha,\frac12}$ splits as the direct sum of two isomorphic parts: $\int_{[0,1)}^{\oplus} V_{\alpha,\frac12,t}dt$ and $\int_{[1,2)}^{\oplus} V_{\alpha,\frac12,t}dt$. This proves (c).

COROLLARY 4.10. *If α satisfies the conditions of Proposition 4.4 with $c/d=\frac12$, then the spectral multiplicity of the transformation $S_{\alpha,\frac12}$ is uniformly equal to two.*

PROBLEM 4.11. Is it possible to conclude, in general, that under the conditions of Proposition 4.4, the spectral multiplicity function of $S_{\alpha,c/d}$ is uniformly equal to d?

REMARK 4.12. From the direct integral decomposition used in the proof of 4.9, one may deduce that if $T_{\alpha,\beta}$ has simple singular spectrum, in particular if α and β satisfy the conditions of Proposition 4.2, then the following holds: there exists a null subset N of $\mathbf{R}$ such that for each $t \in \mathbf{R}\setminus N$, the operator $V_{\alpha,\beta,t}$ has simple, singular spectrum, disjoint from that of any other $V_{\alpha,\beta,t'}$ with $t' \in \mathbf{R}\setminus N$.

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MATHEMATICS DEPARTMENT
THE UNIVERSITY OF WESTERN AUSTRALIA
NEDLANDS, W.A. 6009, AUSTRALIA